End-Semester Exam | ISI Bangalore | May 2018

Introduction to Algebraic Geometry | 100 Points

Notes.
(a) There are a total of 114 points. You can earn a maximum of 100 points.
(b) Justify all your steps. Use only those results proved in class.
(c) By default, k denotes an algebraically closed field.
(d) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.
(e) $\mathbb{A}^n_k$ = the affine n-space over k, $\mathbb{P}^n_k$ = the projective n-space over k.
(f) $\mathcal{V}(-)$ = the common zero locus in suitable affine/projective space of a given collection of polynomials, while $\mathcal{I}(-)$ = the ideal of functions vanishing on a given subset of affine/projective space.
(g) $\mathcal{O}(-)$ = the ring of regular functions on a given quasi-projective algebraic set.

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1. [16 points] Consider the rational map $\phi\colon \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ defined by $[a:b] \mapsto [a^3 : a^2b : ab^2]$.
   (i) Find the locus where ϕ is regular.
   (ii) Show that ϕ extends to a regular map $\overline{\phi}\colon \mathbb{P}^1 \to \mathbb{P}^2$.
   (iii) Show that the image $C = \overline{\phi}(\mathbb{P}^1)$ is a closed subset of $\mathbb{P}^2$.
   (iv) Find a regular map $C \to \mathbb{P}^1$ that is inverse to $\overline{\phi}$.

2. [12 points] Prove that if R is a UFD, then it is integrally closed in its field of fractions. Find the integral closure of $k[t^2, t^3]$ in its field of fractions.

3. [12 points] Let $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ be independent variables over k. Find a transcendental basis of the subfield $F \subset L = k(X_1, \ldots, X_m, Y_1, \ldots, Y_n)$ generated over k by the products X_iY_j.

4. [16 points] Let $C_{a,b}$ be the affine plane curve $\mathcal{V}(y^2 - x(x-a)(x-b))$.
   (i) Verify that if $a, b, 0$ are all distinct, then $C_{a,b}$ is smooth.
   (ii) In the remaining cases, identify the singular points of $C_{a,b}$.

5. [10 points] Let $f\colon X \to Y$ be a map of affine varieties. Show that $\dim(\overline{f(X)}) \le \dim(X)$.

6. [24 points] In this problem we verify that every regular function on projective space is constant. For simplicity, we only consider the case of $\mathbb{P}^1$. Let $k[X_1, X_2]$ be the homogeneous coordinate ring of $\mathbb{P}^1$. For $i = 1, 2$, let U_i be the open set $X_i \neq 0$ and let $U_{12} = U_1 \cap U_2$. Set $T = X_1/X_2$.
   (i) Identify the ring of regular functions $\mathcal{O}[U_i]$ and $\mathcal{O}[U_{12}]$ in terms of T.
   (ii) Now compute $\mathcal{O}[\mathbb{P}^1]$ by identifying pairs (f_1, f_2) with $f_i \in \mathcal{O}[U_i]$ such that the images of f_1 and f_2 in $\mathcal{O}[U_{12}]$ coincide.
   (iii) Deduce that any map $\mathbb{P}^1 \to X$, with X a quasi-affine algebraic set, is a constant map.

7. [24 points] Calculate the intersection multiplicities at the points of intersection of the plane curves $\mathcal{V}(Y^2Z - X^3)$ and $\mathcal{V}(YZ - X^2)$ in $\mathbb{P}^2$.

Hints: To calculate the intersection multiplicity at a point of intersection, work on a suitable affine patch. Simplify the ideal of intersection in the local ring by manipulating the generators and canceling off factors which are units (those that lie outside the maximal ideal). Check if your intersection multiplicities add up to 6.